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A SOLUTION TO AN OPEN PROBLEM ON REVERSE TRIGONOMETRIC MASJED-JAMEI INEQUALITY

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Abstract. In this short note, we prove an open problem for the interval $(-\infty, \infty)$, related to a reverse trigonometric Masjed-Jamei inequality presented in [2] and establish a new inequality of a similar kind.

1. Introduction

In 2010, Masjed-Jamei [3] obtained an upper bound for the square of the inverse tangent function in terms of inverse hyperbolic function. It is formulated as:

$$(\arctan(x))^2 \leq \frac{x \ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} \tag{1.1}$$

holds for all $x \in (-1, 1)$. The right term involves the inverse hyperbolic sine function defined by $\operatorname{arcsinh}(x) = \ln(x+\sqrt{1+x^2})$. Among the recent developments, in 2019, Zhu and Malešević [5] extended the domain of the inequality (1.1) to the whole real line. Precisely, it is stated as

$$(\arctan(x))^2 \leq \frac{x \ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} \tag{1.2}$$

holds for all $x \in (-\infty, \infty)$ and the exponent 2 is the best possible.

In 2021, Chesneau, and Bagul [2] obtained the lower bound for the inverse tangent function involving sine and the inverse hyperbolic sine function. It is stated that:

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For $x \in (-\pi, \pi)$, we have

$$\frac{\sin(x)\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} \leq (\arctan(x))^2 \tag{1.3}$$

and proposed an open problem stated in the following theorem

THEOREM 1.1. *For $x \in (-\infty, \infty)$, we have*

$$\frac{\sin(x)\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} \leq (\arctan(x))^2. \tag{1.4}$$

The first aim of this paper is to prove Theorem 1.1 and the second aim is to prove an inequality analogous to (1.4) which is stated in the following theorem.

THEOREM 1.2. *For $x \in (-1, 1)$, we have*

$$\frac{\tan x \ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} \leq (\arcsin x)^2. \tag{1.5}$$

The inequality (1.5) gives a lower bound for inverse sine function in terms of tangent and the inverse hyperbolic sine function.

2. Proofs of main results

In order to prove our main results, we need the following auxiliary results.

LEMMA 2.1. *For $x > 0$, we have*

$$\sin(x) \quad \leq \quad 2x - \frac{x}{\sqrt{1+x^2}} \tag{2.1}$$

Proof. Let $h(x) = 2x - \frac{x}{\sqrt{1+x^2}} - \sin(x)$. On differentiation we obtain

$$\begin{aligned} h'(x) &= 2 + \frac{x^2}{(1+x^2)^{3/2}} - \frac{1}{\sqrt{1+x^2}} - \cos(x) \\ &= \frac{x^2}{(1+x^2)^{3/2}} + 1 - \frac{1}{\sqrt{1+x^2}} + 1 - \cos(x) \geq 0, \end{aligned}$$

Thus $h(x)$ is increasing and $h(0) \leq h(x)$ implies the inequality (2.1). □

LEMMA 2.2. *If $x \in (0, 1)$ then we have*

$$\arcsin x > \ln(x+\sqrt{1+x^2}). \tag{2.2}$$

Proof. Suppose $k(x) = \arcsin x - \ln(x + \sqrt{1+x^2})$. Then differentiation gives

$$k'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1+x^2}} > 0$$

for $x \in (0,1)$. Therefore $k(x)$ is strictly increasing in $(0,1)$ and the inequality (2.2) follows as $k(x) > k(0) = 0$. □

2.1. Proof of the open problem(Theorem 1.1)

Proof. Let us define

$$g(x) = (\arctan(x))^2 - \frac{\sin(x)\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}$$

We aim to prove that $g(x) \geq 0$ for all $x \in (-\infty, \infty)$. Since $g(-x) = g(x)$, it is enough to prove that $g(x) \geq 0$ for all $x \in (0, \infty)$. Utilizing (2.1) we obtain

$$g(x) \geq (\arctan(x))^2 + \left(\frac{x}{\sqrt{1+x^2}} - 2x\right)\frac{\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}.$$

Define $f(x) = (\arctan(x))^2 + \left(\frac{x}{\sqrt{1+x^2}} - 2x\right)\frac{\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}$. We claim that $f(x) \geq 0$ for all $x \in (0, \infty)$. It is easy to see that $f(0) = 0$. On differentiation we obtain,

$$\begin{aligned} f'(x) =& \frac{1}{(1+x^2)^{5/2}}\Big[(2+2x^2+(x^2-1)\sqrt{1+x^2})\operatorname{arcsinh}(x) \\ &+ (1+x^2)[x(-1+2\sqrt{1+x^2}) + 2\sqrt{1+x^2}\arctan(x)]\Big]. \end{aligned}$$

Define $H_1(x) = 2 + 2x^2 + (x^2-1)\sqrt{1+x^2}$ and $H_2(x) = x(-1 + 2\sqrt{1+x^2})$, then $H_1(0) = 1$ and $H_2(0) = 0$.

$$H_1'(x) = \frac{x + 3x^3 + 4x\sqrt{1+x^2}}{\sqrt{1+x^2}} \geq 0, \quad x \geq 0.$$

Which implies that $H_1(x)$ is increasing and $H_1(x) \geq 1$ for all $x \geq 0$. So $H_2(x) \geq 0$ for all $x \geq 0$. Thus, $f'(x) \geq 0$ for all $x \in (0, \infty)$ implies that f is increasing and $f(x) \geq f(0)$ gives the desired result. □

2.2. Proof of Theorem 1.2

Proof. Clearly, equality holds at $x = 0$. And it is enough to prove inequality (1.5) in $(0,1)$ due to symmetry of functions involved at both sides. Let us first set $G(x) = \sqrt{1+x^2}\arcsin x - \tan x,\ x \in (0,1)$. Differentiation gives

$$\begin{aligned}
G'(x) &= \frac{\sqrt{1+x^2}}{\sqrt{1-x^2}} + \frac{x\arcsin x}{\sqrt{1+x^2}} - \frac{1}{1+x^2} \\
&= \frac{1}{\sqrt{1-x^2}(1+x^2)}\left((1+x^2)^{3/2} + x\sqrt{1-x^4}\arcsin x - \sqrt{1-x^2}\right) \\
&= \frac{1}{\sqrt{1-x^2}(1+x^2)}F(x)
\end{aligned}$$

where

$$F(x) = x\sqrt{1-x^4}\arcsin x + \left((1+x^2)^{3/2} - \sqrt{1-x^2}\right) > 0.$$

Thus $G'(x) > 0$. Hence $G(x)$ is increasing in $(0,1)$ and $G(x) > G(0) = 0$ implies

$$\sqrt{1+x^2}\arcsin x > \tan x,\ x \in (0,1)$$

or

$$\sqrt{1+x^2}(\arcsin x)^2 > \tan x \arcsin x,\ x \in (0,1)$$

Using Lemma 2.2, we get $\tan x \arcsin x > \tan x \ln(x + \sqrt{1+x^2}),\ x \in (0,1)$ which then implies the desired inequality (1.5). □

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